The Fundamental Theorem of Calculus

- Evaluate a definite integral using the Fundamental Theorem of Calculus.
- Understand and use the Mean Value Theorem for Integrals.
- Find the average value of a function over a closed interval.
- Understand and use the Second Fundamental Theorem of Calculus.
- Understand and use the Net Change Theorem.
The Fundamental Theorem of Calculus

Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations. To see how Newton and Leibniz might have anticipated this relationship, consider the approximations shown in Figure 4.26.

The slope of the tangent line was defined using the *quotient* \( \frac{\Delta y}{\Delta x} \).

Similarly, the area of a region under a curve was defined using the *product* \( \Delta y \Delta x \).

So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations.

The Fundamental Theorem of Calculus states that the limit processes preserve this inverse relationship.
**Theorem 4.9** The Fundamental Theorem of Calculus

If a function $f$ is continuous on the closed interval $[a, b]$ and $F$ is an antiderivative of $f$ on the interval $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

The following guidelines can help you understand the use of the Fundamental Theorem of Calculus.

**Guidelines for Using the Fundamental Theorem of Calculus**

1. Provided you can find an antiderivative of $f$, you now have a way to evaluate a definite integral without having to use the limit of a sum.

2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

   $$\int_a^b f(x) \, dx = F(b) - F(a)$$

   For instance, to evaluate $\int_1^3 x^3 \, dx$, you can write

   $$\int_1^3 x^3 \, dx = \left[ \frac{x^4}{4} \right]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$  

3. It is not necessary to include a constant of integration $C$ in the antiderivative because

   $$\int_a^b f(x) \, dx = \left[ F(x) + C \right]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a).$$
Example 1 – Evaluating a Definite Integral

Evaluate each definite integral.

a. \[ \int_{1}^{2} (x^2 - 3) \, dx \]
\[
\left[ \frac{x^3}{3} - 3x \right]_{1}^{2} = \left( \frac{8}{3} - 6 \right) - \left( \frac{1}{3} - 3(1) \right) = \frac{-10}{3} - \frac{8}{3} = \frac{-2}{3}
\]

b. \[ \int_{1}^{4} 3\sqrt{x} \, dx \]
\[
\left[ 3 \cdot \frac{2}{3} x^{\frac{3}{2}} \right]_{1}^{4} = 2 \left( 4^{\frac{3}{2}} - 2 \cdot 1^{\frac{3}{2}} \right) = 2 \left( 8 \right) - 2 = \frac{16 - 2}{4} = \frac{14}{4} = \frac{7}{2}
\]

c. \[ \int_{0}^{\pi/4} \sec^2 x \, dx \]
\[
\left[ \tan x \right]_{0}^{\pi/4} = \tan \frac{\pi}{4} - \tan 0 = 1 - 0 = 1
\]

4.4 notes–solved

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EXAMPLE 2 A Definite Integral Involving Absolute Value

Evaluate $\int_{0}^{2} |2x - 1| \, dx$.

\[
\begin{align*}
&= \int_{0}^{1/2} (2x - 1) \, dx + \int_{1/2}^{2} (1 - 2x) \, dx \\
&= \left[ x^2 - x \right]_{0}^{1/2} + \left[ x - x^2 \right]_{1/2}^{2} \\
&= \left( \frac{1}{4} - \frac{1}{2} \right) - 0 + \left[ (2 - 1) - \left( \frac{1}{4} - \frac{1}{2} \right) \right] \\
&= -\frac{1}{4} + 2 - \frac{1}{4} + \frac{1}{2} \\
&= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2}
\end{align*}
\]
The Mean Value Theorem for Integrals

The area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The Mean Value Theorem for Integrals states that somewhere “between” the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve, as shown in Figure 4.29.

\[ \text{base} \cdot \text{height} = (b-a) \cdot f(c) \]

**Theorem 4.10 Mean Value Theorem for Integrals**

If \( f \) is continuous on the closed interval \([a, b]\), then there exists a number \( c \) in the closed interval \([a, b]\) such that

\[ \int_a^b f(x) \, dx = f(c)(b - a). \]

**Note** Notice that Theorem 4.10 does not specify how to determine \( c \). It merely guarantees the existence of at least one number \( c \) in the interval.
Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of $f$ on the interval $[a, b]$.

### Definition of the Average Value of a Function on an Interval

If $f$ is integrable on the closed interval $[a, b]$, then the average value of $f$ on the interval is

$$
\frac{1}{b-a} \int_a^b f(x) \, dx = f(c)
$$

---

**Example 4 – Finding the Average Value of a Function**

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

$$
\frac{1}{4-1} \int_1^4 (3x^2 - 2x) \, dx
$$

$$
\frac{1}{3} \int_1^4 (3x^2 - 2x) \, dx
$$

$$
\frac{1}{3} \left[ x^3 - x^2 \right]_1^4
$$

$$
\frac{1}{3} \left[ (4^3 - 4^2) - (1^3 - 1^2) \right]
$$

$$
\frac{1}{3} \left[ (64 - 16 - 0) \right]
$$

$$
\frac{1}{3} \left[ 48 \right]
$$

$$
\frac{16}{3}
$$
The Second Fundamental Theorem of Calculus

The definite integral of \( f \) on the interval \([a, b]\) is defined using the constant \( b \) as the upper limit of integration and \( x \) as the variable of integration.

A slightly different situation may arise in which the variable \( x \) is used in the upper limit of integration.

To avoid the confusion of using \( x \) in two different ways, \( t \) is temporarily used as the variable of integration.

**Example 6 – The Definite Integral as a Function**

Evaluate the function

\[
F(x) = \int_0^x \cos t \, dt
\]

at \( x = 0, \pi/6, \pi/4, \pi/3, \) and \( \pi/2. \)

\[
F(x) = \sin t \bigg|_0^x = \sin x - \sin 0 = \sin x
\]

\[
what \ is \ F'(x) ?
\]

\[
F'(x) = \frac{d}{dx} \left( \int_0^x \cos t \, dt \right)
\]

\[
f'(x)(x) = \cos x
\]

\[
f'(x)(x) = \cos x
\]

\[
F'(x) = \cos x
\]

Find the derivative of

\[
F(x) = \int_0^x (\cos t + \alpha) \, dt
\]

\[
F'(x) = \frac{d}{dx} \left( \int_0^x (\cos t + \alpha) \, dt \right)
\]

\[
= (\cos (x) + \alpha) - (\cos (0) + \alpha)
\]

\[
= 3x^2 \cos x^2 - 0
\]

\[
F'(x) = 3x^2 \cos x^2
\]

Find \( F'(x) \), given \( F(x) = \int_x^{x+1} t^3 \, dt \)

\[
F'(x) = \frac{d}{dx} \left( \int_x^{x+1} t^3 \, dt \right)
\]

\[
= (t^4 + 1) \bigg|_x^{x+1} = (4(x+1)^4 + 1) - (4x^4 + 1)
\]

\[
= 4x^4 + 8 + 1 - 4x^4
\]

\[
= 4x^4 + 9 - 4x^4
\]

\[
F'(x) = 8
\]

\[
F(x) = \int_0^x t^3 \, dt = 0
\]

\[
F'(x) = \frac{d}{dx} \left( \int_0^x t^3 \, dt \right)
\]

\[
= t^3(1) - (-x)^3(-1)
\]

\[
= x^3 + x^3
\]

\[
= 0
\]
**Net Change Theorem**

The Fundamental Theorem of Calculus states that if $f$ is continuous on the closed interval $[a, b]$ and $F$ is an antiderivative of $f$ on $[a, b]$, then

$$
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
$$

But because $F'(x) = f(x)$, this statement can be rewritten as

$$
\int_{a}^{b} F'(x) \, dx = F(b) - F(a)
$$

where the quantity $F(b) - F(a)$ represents the *net change of $F$ on the interval* $[a, b]$.

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**THEOREM 4.12 THE NET CHANGE THEOREM**

The definite integral of the rate of change of a quantity $F'(x)$ gives the total change, or *net change*, in that quantity on the interval $[a, b]$.

$$
\int_{a}^{b} F'(x) \, dx = F(b) - F(a) \quad \text{Net change of } F
$$
Example 9 – Using the Net Change Theorem

A chemical flows into a storage tank at a rate of $180 + 3t$ liters per minute, where $0 \leq t \leq 60$. Find the amount of the chemical that flows into the tank during the first 20 minutes.

\[
\int_0^{20} (180 + 3t) \, dt \\
\left[ 180t + \frac{3t^2}{2} \right]_0^{20} \\
180(20) + \frac{3(20)^2}{2} - \left( 0 + 0 \right) \\
3600 + 600 \\
4200 \text{ liters of chemical flowed into the tank during the first 20 minutes.}
\]
Net Change Theorem

The velocity of a particle moving along a straight line where \( s(t) \) is the position at time \( t \). Then its velocity is \( v(t) = s'(t) \) and

\[
\int_{a}^{b} v(t) \, dt = s(b) - s(a).
\]

This definite integral represents the net change in position, or displacement, of the particle.

When calculating the total distance traveled by the particle, you must consider the intervals where \( v(t) \leq 0 \) and the intervals where \( v(t) \geq 0 \).

When \( v(t) \leq 0 \) the particle moves to the left, and when \( v(t) \geq 0 \), the particle moves to the right.

To calculate the total distance traveled, integrate the absolute value of velocity \( |v(t)| \).

So, the displacement of a particle and the total distance traveled by a particle over \([a, b]\) can be written as

\[
\text{Displacement on } [a, b] = \int_{a}^{b} v(t) \, dt = A_1 - A_2 + A_3
\]

\[
\text{Total distance traveled on } [a, b] = \int_{a}^{b} |v(t)| \, dt = A_1 + A_2 + A_3
\]

(see Figure 4.36).

\[A_1, A_2, \text{ and } A_3 \text{ are the areas of the shaded regions.}\]
Example 10 – Solving a Particle Motion Problem

A particle is moving along a line so that its velocity is \( v(t) = t^3 - 10t^2 + 29t - 20 \) feet per second at time \( t \).

a. What is the displacement of the particle on the time interval \( 1 \leq t \leq 5 \)?

\[
\int_1^5 v(t) \, dt = \int_1^5 (t^3 - 10t^2 + 29t - 20) \, dt
\]

\[
\left[ \frac{1}{4} t^4 - \frac{10}{3} t^3 + \frac{29}{2} t^2 - 20t \right]_1^5
\]

\[
= \left( \frac{1}{4}(5^4) - \frac{10}{3}(5^3) + \frac{29}{2}(5^2) - 20(5) \right) - \left( \frac{1}{4}(1^4) - \frac{10}{3}(1^3) + \frac{29}{2}(1^2) - 20(1) \right)
\]

\[
= \left( \frac{625}{4} - \frac{1250}{3} + \frac{725}{2} - 100 \right) - \left( \frac{1}{4} - \frac{10}{3} + \frac{29}{2} - 20 \right)
\]

\[
= \frac{625}{4} - \frac{1250}{3} + \frac{725}{2} - 100 - \left( \frac{1}{4} - \frac{10}{3} + \frac{29}{2} - 20 \right)
\]

\[
= \frac{625 - 500 + 1450}{4} - \frac{1250 - 400 + 560}{3} + \frac{725 - 140 + 290}{2} - 100 - \left( \frac{1}{4} - \frac{10}{3} + \frac{29}{2} - 20 \right)
\]

\[
= \frac{650}{4} - \frac{350}{3} + \frac{1075}{2} - 100 - \left( \frac{1}{4} - \frac{10}{3} + \frac{29}{2} - 20 \right)
\]

\[
= \frac{650}{4} - \frac{350}{3} + \frac{1075}{2} - 100 - \frac{1}{4} + \frac{10}{3} - \frac{29}{2} + 20
\]

\[
= \frac{650 - 140 + 1075 - 400}{4} - \frac{350 - 120 + 560 - 120}{3} + \frac{1075 - 140 + 290}{2} - 100 - \frac{1}{4} + \frac{10}{3} - \frac{29}{2} + 20
\]

\[
= \frac{1300 - 600}{4} - \frac{850 - 360}{3} + \frac{1225 - 140 + 290}{2} - 100 - \frac{1}{4} + \frac{10}{3} - \frac{29}{2} + 20
\]

\[
= \frac{700}{4} - \frac{490}{3} + \frac{1375}{2} - 100 - \frac{1}{4} + \frac{10}{3} - \frac{29}{2} + 20
\]

\[
= \frac{350}{2} - \frac{490}{3} + \frac{1375}{2} - 100 - \frac{1}{4} + \frac{10}{3} - \frac{29}{2} + 20
\]

\[
= \frac{350 - 245 + 1375 - 400}{2} - \frac{1}{4} + \frac{10}{3} - \frac{29}{2} + 20
\]

\[
= \frac{1250}{2} - \frac{1}{4} + \frac{10}{3} - \frac{29}{2} + 20
\]

\[
= \frac{625}{1} - \frac{1}{4} + \frac{10}{3} - \frac{29}{2} + 20
\]

\[
= \frac{625 - 1 + 10 - 58 + 80}{4}
\]

\[
= \frac{664}{4}
\]

\[
= \frac{166}{1}
\]

\[
= 166
\]

b. What is the total distance traveled by the particle on the time interval \( 1 \leq t \leq 5 \)?

\[
v(t) = t^3 - 10t^2 + 29t - 20
\]

\[
\begin{array}{c|c|c|c}
\text{t} & 1 & 5 & 0 \\
\hline
v(t) & t^3 - 10t^2 + 29t - 20 & t^2 - 9t + 20 & 0 \\
\hline
\end{array}
\]

\[
\int_1^5 |v(t)| \, dt = \int_1^5 (t^3 - 10t^2 + 29t - 20) \, dt
\]

\[
\int_1^5 (t^3 - 10t^2 + 29t - 20) \, dt = \int_1^5 v(t) \, dt + \int_1^5 -v(t) \, dt
\]

\[
= \int_1^5 v(t) \, dt + \int_1^5 v(t) \, dt
\]

\[
= 2 \int_1^5 v(t) \, dt
\]

\[
= 2 \times \frac{166}{1}
\]

\[
= 332
\]
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