Riemann Sums and Definite Integrals

- Evaluate a definite integral using properties of definite integrals.
Definite Integrals

**THEOREM 4.4 CONTINUITY IMPLIES INTEGRABILITY**

If a function $f$ is continuous on the closed interval $[a, b]$, then $f$ is integrable on $[a, b]$. That is, $\int_{a}^{b} f(x) \, dx$ exists.

Definite integrals can be positive, negative, or zero.

**THEOREM 4.5 THE DEFINITE INTEGRAL AS THE AREA OF A REGION**

If $f$ is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of $f$, the $x$-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_{a}^{b} f(x) \, dx.$$  

(See Figure 4.21.)

Because $f$ is continuous and nonnegative on the closed interval $[0, 4]$, the area of the region is

$$\text{Area} = \int_{0}^{4} (4x - x^2) \, dx.$$ 

**Evaluate the definite integral**

$$\int_{-2}^{1} 2x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \left( -2 + \frac{3}{n} \right) \left( \frac{3}{n} \right) = \lim_{n \to \infty} \frac{6}{n} \sum_{i=1}^{n} \left( -2 + \frac{3}{n} \right) = \lim_{n \to \infty} \frac{6}{n} \left[ -2n + \frac{3}{n} \left( \frac{nn + 1}{2} \right) \right] = \lim_{n \to \infty} \left( -12 + 9 + \frac{9}{n} \right) = -3.$$ 

Because the definite integral is negative, it does not represent the area of the region.
Definite Integrals

You can evaluate a definite integral in two ways—you can use the limit definition or you can check to see whether the definite integral represents the area of a common geometric region such as a rectangle, triangle, or semicircle.

Set up a definite integral that yields the area of the region.

\[ \int_{1}^{3} 4 \, dx \]

\[ \int_{0}^{3} (x + 2) \, dx \]

\[ \int_{-2}^{2} \sqrt{4 - x^2} \, dx \]
Properties of Definite Integrals

**DEFINITIONS OF TWO SPECIAL DEFINITE INTEGRALS**

1. If \( f \) is defined at \( x = a \), then we define \( \int_a^a f(x) \, dx = 0 \).

2. If \( f \) is integrable on \([a, b]\), then we define \( \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \).

**Example 4 – Evaluating Definite Integrals**

a. Because the sine function is defined at \( x = \pi \), and the upper and lower limits of integration are equal, you can write

\[
\int_0^\pi \sin x \, dx = 0.
\]

b. The integral \( \int_{\frac{3}{2}}^1 (x + 2) \, dx \) has a value of \( \frac{21}{2} \), you can write

\[
\int_{\frac{3}{2}}^1 (x + 2) \, dx = -\int_1^{\frac{3}{2}} (x + 2) \, dx = -\frac{21}{2}.
\]
Properties of Definite Integrals

**Theorem 4.6 Additive Interval Property**
If \( f \) is integrable on the three closed intervals determined by \( a, b, \) and \( c, \) then

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]

![Diagram showing additive property of definite integrals](image)

**Example 5 – Using the Additive Interval Property**

\[
\int_{-1}^{1} |x| \, dx = \int_{-1}^{0} x \, dx + \int_{0}^{1} x \, dx
\]

\[
= \left[ \frac{1}{2} x^2 \right]_{-1}^{0} + \left[ \frac{1}{2} x^2 \right]_{0}^{1}
\]

\[
= \left( \frac{1}{2} \right) + \left( \frac{1}{2} \right)
\]

\[
= 1
\]

Area \( \Delta = \frac{1}{2}bh \)
Properties of Definite Integrals

THEOREM 4.7 PROPERTIES OF DEFINITE INTEGRALS

If \( f \) and \( g \) are integrable on \([a, b]\) and \( k \) is a constant, then the functions \( kf \) and \( f \pm g \) are integrable on \([a, b]\), and

1. \( \int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx \)
2. \( \int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \).

Note that Property 2 of Theorem 4.7 can be extended to cover any finite number of functions. For example,

\[
\int_a^b [f(x) + g(x) + h(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx + \int_a^b h(x) \, dx.
\]

Example 6 – Evaluation of a Definite Integral

Evaluate \( \int_0^1 (-x^2 + 4x - 3) \, dx \) using each of the following values.

\[
\int_0^1 x^2 \, dx = \frac{2}{3}, \quad \int_0^3 x \, dx = 4, \quad \int_1^3 1 \, dx = 2
\]

\[
\int_0^3 -x^2 \, dx + \int_1^3 4x \, dx - \int_1^3 3 \, dx
\]

\[
-\int_1^3 x^2 \, dx + 4\int_1^3 x \, dx - 3\int_1^3 1 \, dx
\]

\[
= -\frac{26}{3} + 4(4) - 3(2)
\]

\[
= -\frac{26}{3} + 16 - 6
\]

\[
= -\frac{26}{3} + 10
\]

\[
= -\frac{26}{3} + \frac{30}{3} = \frac{4}{3}
\]

\[
p. 278
\]

13 – 21 odd
53 – 41 odd

46