Differentials

- Understand the concept of a tangent line approximation.
- Compare the value of the differential, \( dy \), with the actual change in \( y, \Delta y \).
- Find the differential of a function using differentiation formulas.
Tangent Line Approximations

What is the point slope formula?

\[
\begin{align*}
    y - y_1 &= m(x - x_1) \\
    y - f(c) &= m(x - c) \\
    y - f(c) &= f'(c)(x - c)
\end{align*}
\]

\[
y = f(c) + f'(c)(x - c)
\]

is called the tangent line approximation (or linear approximation) of \( f \) at \( c \).

Example 1 – Using a Tangent Line Approximation

Find the tangent line approximation of

\[ f(x) = 1 + \sin x \]

at the point \((0, 1)\). Then use a table to compare the \( y \)-values of the linear function with those of \( f(x) \) on an open interval containing \( x = 0 \).

\[
\begin{align*}
    y - y_1 &= m(x - x_1) \\
    y - 1 &= m(x - 0) \\
    y - 1 &= 1(x - 0) \\
    y - 1 &= 1(x) \\
    y - 1 &= x \\
    y &= x + 1
\end{align*}
\]

\[
\begin{align*}
    f'(x) &= \cos x \\
    f'(0) &= \cos(0) \\
    &= 1
\end{align*}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x & -0.5 & -0.1 & -0.01 & 0 & 0.01 & 0.1 & 0.5 \\
\hline
f(x) = 1 + \sin x & 0.521 & 0.9002 & 0.990002 & 1 & 1.009998 & 1.0998 & 1.479 \\
\hline
y = 1 + x & 0.5 & 0.9 & 0.99 & 1 & 1.01 & 1.1 & 1.5 \\
\hline
\end{array}
\]
Differentials

When the tangent line to the graph of $f$ at the point $(c, f(c))$

$$y = f(c) + f'(c)(x - c)$$

is used as an approximation of the graph of $f$, the quantity $x - c$ is called the change in $x$, and is denoted by $\Delta x$, as shown in Figure 3.66.

$$\Delta y = f(c + \Delta x) - f(c) \approx f'(c)\Delta x$$

For such an approximation, the quantity $\Delta x$ is traditionally denoted by $dx$, and is called the differential of $x$.

The expression $f'(x)dx$ is denoted by $dy$, and is called the differential of $y$.  

$dy = f'(x)dx$
Example 2 – Comparing $\Delta y$ and $dy$

Let $y = x^2$. Find $dy$ when $x = 1$ and $dx = 0.01$.

Compare this value with $\Delta y$ for $x = 1$ and $\Delta x = 0.01$.

\[
\begin{align*}
\Delta y &= f(c+\Delta x) - f(c) \\
        &= f(1.01) - f(1) \\
        &= 1.01^2 - 1^2 \\
        &= 1.0201 - 1 \\
        &= 0.0201
\end{align*}
\]

\[
\begin{align*}
dy &= f'(x)\,dx \\
    &= f'(1)\cdot 0.01 \\
    &= 2 \cdot 0.01 \\
    &= 0.02
\end{align*}
\]

Almost the same.
## Example 4 – Finding Differentials

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
<th>Differential</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( y = x^2 )</td>
<td>( \frac{dy}{dx} = 2x )</td>
<td>( dy = 2x , dx )</td>
</tr>
<tr>
<td>b. ( y = 2 \sin x )</td>
<td>( \frac{dy}{dx} = 2 \cos x )</td>
<td>( dy = 2 \cos x , dx )</td>
</tr>
<tr>
<td>c. ( y = x \cos x )</td>
<td>( \frac{dy}{dx} = -x \sin x + \cos x )</td>
<td>( dy = (-x \sin x + \cos x) , dx )</td>
</tr>
<tr>
<td>d. ( y = \frac{1}{x} )</td>
<td>( \frac{dy}{dx} = -\frac{1}{x^2} )</td>
<td>( dy = -\frac{dx}{x^2} )</td>
</tr>
<tr>
<td>5 ( = x^{-1} )</td>
<td></td>
<td>( dy = -\frac{1}{x} , dx )</td>
</tr>
</tbody>
</table>
Calculating Differentials

Differentials can be used to approximate function values. To do this for the function given by \( y = f(x) \), use the formula

\[
f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x) \Delta x
\]

Example 7 – Approximating Function Values

Use differentials to approximate \( \sqrt{16.5} \).

\[
\begin{align*}
\sqrt{16.5} &= \sqrt{16 + 0.5} \\
&= 4 + \frac{d}{dx} \sqrt{x} \bigg|_{x=16} \cdot 0.5 \\
&= 4 + \frac{1}{2 \sqrt{x}} \bigg|_{x=16} \cdot 0.5 \\
&= 4 + \frac{1}{2 \cdot 4} \cdot 0.5 \\
&= 4 + 0.125 \\
&= 4.125
\end{align*}
\]

Therefore, \( \sqrt{16.5} \approx 4.125 \).