3.2 Rolle’s Theorem and the Mean Value Theorem

- Understand and use Rolle’s Theorem.
- Understand and use the Mean Value Theorem.
Rolle’s Theorem

**THEOREM 3.3 ROLLE’S THEOREM**

Let \( f \) be continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\). If \( f(a) = f(b) \), then there is at least one number \( c \) in \((a, b)\) such that \( f'(c) = 0 \).

From Rolle’s Theorem, you can see that if a function \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), and if \( f(a) = f(b) \), there must be at least one \( x \)-value between \( a \) and \( b \) at which the graph of \( f \) has a horizontal tangent, as shown in Figure 3.8(a).

If the differentiability requirement is dropped from Rolle’s Theorem, \( f \) will still have a critical number in \((a, b)\), but it may not yield a horizontal tangent. Such a case is shown in Figure 3.8(b).
Example 1 – Illustrating Rolle’s Theorem

Find the two x-intercepts of

\[ f(x) = x^2 - 3x + 2 \]

and show that \( f'(x) = 0 \) at some point between the two x-intercepts.

\[ 0 = x^2 - 3x + 2 \]
\[ (x - 2)(x - 1) = 0 \]
\[ x = 2 \quad \text{or} \quad x = 1 \]

\[ f'(x) = 2x - 3 \]
\[ 2x - 3 = 0 \]
\[ 2x = 3 \]
\[ x = \frac{3}{2} = 1.5 \]

The x-value for which \( f'(x) = 0 \) is between the two x-intercepts.
EXAMPLE 12 Illustrating Rolle’s Theorem

Let \( f(x) = x^4 - 2x^2 \). Find all values of \( c \) in the interval \((-2, 2)\) such that \( f'(c) = 0 \).

Polynomial
- continuous
- smooth around
- differentiable

\[ f(-2) = (-2)^4 - 2(-2)^2 = 8 \]
\[ f(2) = 2^4 - 2(2)^2 = 8 \]

Rolle’s Applies!

\[ f'(x) = 4x^3 - 4x \]
\[ 4x^3 - 4x = 0 \]
\[ 4x(x^2 - 1) = 0 \]
\[ 4x = 0 \quad x^2 - 1 = 0 \]
\[ x = 0 \quad (x - 1)(x + 1) = 0 \]
\[ x = 1 \quad x = -1 \]

\( f'(x) = 0 \) for more than one \( x \)-value in the interval \((-2, 2)\).
The Mean Value Theorem

Rolle’s Theorem can be used to prove another theorem—the **Mean Value Theorem**.

**THEOREM 3.4 THE MEAN VALUE THEOREM**

If \( f \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\), then there exists a number \( c \) in \((a, b)\) such that

\[
 f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

What are some of the implications of the Mean Value Theorem?

- It guarantees the existence of a tangent line that is parallel to the secant line through \((a, f(a))\) and \((b, f(b))\).

- It also guarantees that there is a time when the instantaneous rate of change is equal to the average rate of change over the interval \([a, b]\).
Example 3 – Finding a Tangent Line

Given \( f(x) = 5 - \frac{4}{x} \), find all values of \( c \) in the open interval \((1, 4)\) such that

\[
f'(c) = \frac{f(4) - f(1)}{4 - 1}.
\]

\[
f(4) = 5 - \frac{4}{4} = 4
\]

\[
f(1) = 5 - \frac{4}{1} = 1
\]

\[
= \frac{4 - 1}{4 - 1} = \frac{3}{1} = 3
\]

\[\text{Prop. of secant line is 3} \]

So, where does \( f'(c) = 1 \)?

\[
f(x) = 5 - \frac{4}{x}
\]

\[
= 5 - 4x^{-1}
\]

\[
f'(x) = 4x^{-2}
\]

\[
4x^{-2} = 1
\]

\[
\frac{4}{x^2} = 1
\]

\[
x^2 = 4
\]

\[
x = \pm 2
\]

since the interval is \((1, 4)\), 

-2 does not meet the criteria.

\[\therefore \text{At } x = 2, \text{ the slope of the tangent line is the same as the slope of the secant line through the interval endpoints.}\]

Example 4 Finding an Instantaneous Rate of Change

Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in Figure 3.14. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) at some time during the 4 minutes.

\[a = r \cdot t\]

\[5 = r \cdot \frac{4}{15}\]

\[r = 75 \text{ mph}\]

\[\text{Yes. He exceeded the 55 mph speed limit.}\]
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