Related Rates

- Find a related rate.
- Use related rates to solve real-life problems.
Finding Related Rates

The Chain Rule can be used to find \( \frac{dy}{dx} \) implicitly. Another important use of the Chain Rule is to find the rates of change of two or more related variables that are changing with respect to time.

For example, when water is drained out of a conical tank (see Figure 2.33), the volume \( V \), the radius \( r \), and the height \( h \) of the water level are all functions of time \( t \).

Knowing that these variables are related by the equation

\[
V = \frac{\pi}{3} r^2 h
\]

you can differentiate implicitly with respect to \( t \) to obtain the related-rate equation

\[
\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)
\]

\[
\frac{dV}{dt} = \frac{\pi}{3} \left[ r^2 \frac{dh}{dt} + h \left( 2r \frac{dr}{dt} \right) \right]
\]

\[
= \frac{\pi}{3} \left( r^2 \frac{dh}{dt} + 2rh \frac{dr}{dt} \right).
\]

From this equation you can see that the rate of change of \( V \) is related to the rates of change of both \( h \) and \( r \).
Example 1 – Two Rates That Are Related

Suppose $x$ and $y$ are both differentiable functions of $t$ and are related by the equation $y = x^2 + 3$.

Find $\frac{dy}{dt}$ when $x = 1$, given that $\frac{dx}{dt} = 2$ when $x = 1$.

$$y = x^2 + 3$$

$$\frac{dy}{dx} = 2x \frac{dx}{dt}$$

$$\frac{dy}{dt} = 2 \cdot 1 \cdot 2$$

$$\frac{dy}{dt} = 4$$
5) \( y = 4(x^2 - 5x) \)
\[
\frac{dy}{dx} = 4 \left( 2x \frac{dx}{dx} - 5 \frac{dx}{dx} \right) \\
\uparrow \\
\quad S = 4(2x - 5) \\
\quad S = 8 \frac{dx}{dx} - 20 \frac{dx}{dx} \\
\quad S = -12 \frac{dx}{dx} \\
\frac{dx}{dt} = \frac{5}{12} \]
Problem Solving with Related Rates

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.

2. Write an equation involving the variables whose rates of change either are given or are to be determined.

3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time* $t$.

4. After completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.
Example 3 – An Inflating Balloon

Air is being pumped into a spherical balloon (see Figure 2.35) at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.

\[
\frac{dV}{dt} = 4.5
\]

Find \( \frac{dr}{dt} \) when \( r = 2 \)

\[
V = \frac{4}{3} \pi r^3
\]

\[
\frac{dV}{dt} = \frac{4}{3} \pi \cdot 3r^2 \frac{dr}{dt}
\]

\[
\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}
\]

\[
4.5 = 4\pi \cdot 2^2 \cdot \frac{dr}{dt}
\]

\[
4.5 = 16\pi \cdot \frac{dr}{dt}
\]

\[
\frac{4.5}{16\pi} = \frac{dr}{dt}
\]

\[
\frac{dr}{dt} \approx 0.09 \text{ ft/min}
\]
wk 2 in west

given: \( \frac{dr}{dt} = 1 \)

problem: \( \frac{dA}{dt} \) when \( r = 4 \)

\[
A = \pi r^2
\]

\[
\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}
\]

\[
\frac{dA}{dt} = \pi \cdot 2 \cdot 4 \cdot 1
\]

\[
\frac{dA}{dt} = 8\pi
\]

When \( r = 4 \), the area is changing at a rate of \( 8\pi \text{ square units per second} \).
Example 4: The Speed of an Airplane Tracked by Radar

An airplane is flying on a flight path that will take it directly over a radar tracking station, as shown in Figure 2.36. If $s$ is decreasing at a rate of 400 miles per hour when $s = 10$ miles, what is the speed of the plane?

Find \( \frac{dx}{dt} \) when $s = 10$ (also $x = 8$).

\[
\begin{align*}
\frac{ds}{dt} &= -400 \\
\frac{dx}{dt} &= \frac{2s}{x} \frac{ds}{dt} \\
\frac{dx}{dt} &= \frac{2 \cdot 10}{8} (-400) \\
\frac{dx}{dt} &= -500 \\
\end{align*}
\]

500 mph
154
1-9 odd
13, 17, 21, 25, 33