2.2 Basic Differentiation Rules and Rates of Change

- Find the derivative of a function using the Constant Rule.

- Find the derivative of a function using the Power Rule.

- Find the derivative of a function using the Constant Multiple Rule.

- Find the derivative of a function using the Sum and Difference Rules.

- Find the derivatives of the sine function and of the cosine function.

- Use derivatives to find rates of change.
The Constant Rule

**THEOREM 2.2 THE CONSTANT RULE**

The derivative of a constant function is 0. That is, if $c$ is a real number, then

$$\frac{d}{dx}[c] = 0.$$

(See Figure 2.14.)

---

**Example 1 – Using the Constant Rule**

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $y = 7$</td>
<td>$\frac{dy}{dx} = 0$</td>
</tr>
<tr>
<td>b. $f(x) = 0$</td>
<td>$f'(x) = 0$</td>
</tr>
<tr>
<td>c. $s(t) = -3$</td>
<td>$s'(t) = 0$</td>
</tr>
<tr>
<td>d. $y = k\pi^2$, $k$ is constant</td>
<td>$y' = 0$</td>
</tr>
</tbody>
</table>
The Power Rule

**Theorem 2.3: The Power Rule**

If \( n \) is a rational number, then the function \( f(x) = x^n \) is differentiable and

\[
\frac{d}{dx}[x^n] = nx^{n-1}.
\]

For \( f \) to be differentiable at \( x = 0 \), \( n \) must be a number such that \( x^{n-1} \) is defined on an interval containing 0.

Does this rule work for \( f(x) = x^1 \)? *Yes!*

\[
f'(x) = 1 \cdot x^0 = 1
\]

**Example 2 – Using the Power Rule**

a. \( f(x) = x^3 \)
   \[
f'(x) = 3 \cdot x^2
   \]

b. \( g(x) = \sqrt[3]{x} \)
   \[
g'(x) = \frac{1}{3} \cdot x^{-\frac{2}{3}}
   \]

c. \( y = \frac{1}{x^2} \)
   \[
y' = -2 \cdot x^{-3}
   \]

In Example 2(c), note that before differentiating, \( 1/x^2 \) was rewritten as \( x^{-2} \). Rewriting is the first step in many differentiation problems.
**EXAMPLE 3** Finding the Slope of a Graph

Find the slope of the graph of \( f(x) = x^4 \) when

- \( x = -1 \)
- \( x = 0 \)
- \( x = 1 \)

\[
\begin{align*}
 f'(x) &= 4x^3 \\
 f'(-1) &= 4(-1)^3 = -4 \\
 f'(0) &= 0 \\
 f'(1) &= 4 \\
 f'(-1) &= -4 
\end{align*}
\]

**EXAMPLE 4** Finding an Equation of a Tangent Line

Find an equation of the tangent line to the graph of \( f(x) = x^2 \) when \( x = -2 \).

\[
\begin{align*}
 f'(x) &= 2x \\
 f'(-2) &= 2(-2) = -4 \\
 \text{eq:} \quad y - y_1 &= m(x - x_1) \\
 \text{slope:} \quad m &= -4 \\
 \text{point:} \quad (-2, 4) \\
 f(-2) &= (-2)^2 = 4 \\
 y &= -4(x + 2) \\
 y &= -4x - 8 \\
 y &= -4y - 4 
\end{align*}
\]
The Constant Multiple Rule

**THEOREM 2.4 THE CONSTANT MULTIPLE RULE**

If $f$ is a differentiable function and $c$ is a real number, then $cf$ is also differentiable and \( \frac{d}{dx}[cf(x)] = cf'(x) \).

**Example 5 – Using the Constant Multiple Rule**

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>$y = \frac{2}{x}$</td>
<td>b.</td>
<td>$f(t) = \frac{4t^2}{5}$</td>
<td>c.</td>
</tr>
<tr>
<td></td>
<td>$y = 2 \cdot \frac{1}{x}$</td>
<td></td>
<td>$f'(t) = \frac{8}{5} \cdot 2t$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y' = 2 \cdot \frac{1}{x^2}$</td>
<td></td>
<td>$y' = \frac{8}{5}t$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y'' = -2x^{-3}$</td>
<td></td>
<td>$y'' = \frac{8}{5}$</td>
<td></td>
</tr>
<tr>
<td>e.</td>
<td>$y = -\frac{3x^4}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y' = -\frac{3}{2} \cdot 4x^3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y'' = 3 \cdot \frac{15}{2} x^2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y''' = 3 \cdot \frac{45}{2} x$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**more examples!!!**

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>$y = \frac{5}{2x^3}$</td>
<td>b.</td>
<td>$y = \frac{5}{(2x)^3}$</td>
<td>c.</td>
</tr>
<tr>
<td></td>
<td>$y = \frac{5x^{-3}}{2}$</td>
<td></td>
<td>$y = \frac{5x^{-3}}{8}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y' = -15 \cdot \frac{x^{-4}}{2}$</td>
<td></td>
<td>$y' = -15 \cdot \frac{x^{-3}}{8}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y'' = -15 \cdot \frac{x^{-5}}{2}$</td>
<td></td>
<td>$y'' = -15 \cdot \frac{x^{-4}}{8}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y''' = -15 \cdot \frac{x^{-6}}{2}$</td>
<td></td>
<td>$y''' = -15 \cdot \frac{x^{-5}}{8}$</td>
<td></td>
</tr>
<tr>
<td>e.</td>
<td>$y = \frac{7}{(3x)^{-2}}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y = 7 \cdot \frac{1}{(3x)^2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y = 7 \cdot \frac{9}{3x^2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y = 63 \cdot x^2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$y' = 126 \cdot x$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The Sum and Difference Rules

**THEOREM 2.5 THE SUM AND DIFFERENCE RULES**

The sum (or difference) of two differentiable functions $f$ and $g$ is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of $f$ and $g$.

\[
\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \quad \text{Sum Rule}
\]

\[
\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \quad \text{Difference Rule}
\]

**Example 7 – Using the Sum and Difference Rules**

**a.** $f(x) = x^3 - 4x + 5$

\[f'(x) = 3x^2 - 4\]

\[f'(x) = 3x^2 - 4\]

\[f'(x) = 3x^2 - 4\]

**b.** $g(x) = \frac{-x^4}{2} + 3x^3 - 2x$

\[g'(x) = -2x^3 + 9x^2 - 2\]
Derivatives of the Sine and Cosine Functions

**Theorem 2.6** Derivatives of Sine and Cosine Functions

\[
\frac{d}{dx} \sin x = \cos x \quad \quad \frac{d}{dx} \cos x = -\sin x
\]

Example 8 – Derivatives Involving Sines and Cosines

- **a.** \( y = 2 \sin x \)
  \[ y' = 2\cos x \]

- **b.** \( y = \frac{1}{2} \sin x \)
  \[ y' = \frac{1}{2} \cos x \]

- **c.** \( y = x + \cos x \)
  \[ y' = 1 - \sin x \]

\[ 0.115 \quad 1 - 6 \cos \theta \]
Rates of Change

You have seen how the derivative is used to determine slope.

The derivative can also be used to determine the rate of change of one variable with respect to another.

Applications involving rates of change occur in a wide variety of fields.

A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

average velocity is

\[
\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}.
\]

Example 9 – Finding Average Velocity of a Falling Object

If a billiard ball is dropped from a height of 100 feet, its height \( s \) at time \( t \) is given by the position function

\[ s = -16t^2 + 100 \]

where \( s \) is measured in feet and \( t \) is measured in seconds.

Find the average velocity over each of the following time intervals.

\[
\begin{align*}
\text{a.} \ [1, 2] & \quad (1, 184) \quad (2, 36) \\
&s(1) = -16 + 100 \\
&s(2) = -64 + 100 \\
&\Delta s = -48 \quad \Delta t = 1 \\
&\text{Average Velocity} = \frac{-48}{1} = -48 \text{ ft/s}
\end{align*}
\]

\[
\begin{align*}
\text{b.} \ [1, 1.5] & \quad (1, 184) \quad (1.5, 64) \\
&s(1) = -36 + 100 \\
&s(1.5) = 64 \\
&\Delta s = 28 \quad \Delta t = 0.5 \\
&\text{Average Velocity} = \frac{28}{0.5} = 56 \text{ ft/s}
\end{align*}
\]

\[
\begin{align*}
\text{c.} \ [1, 1.1] & \quad (1, 184) \quad (1.1, 80.64) \\
&s(1) = 80.64 \\
&s(1.1) = 80.64 \\
&\Delta s = 0 \quad \Delta t = 0.1 \\
&\text{Average Velocity} = \frac{0}{0.1} = 0 \text{ ft/s}
\end{align*}
\]

Note that the average velocities are negative, indicating that the object is moving downward.

Velocity can be negative, zero, or positive.

The speed of an object is the absolute value of its velocity.

Speed cannot be negative.
Instantaneous velocity

\[ v(t) = \lim_\limits{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t). \]

What is the difference (besides the formulas!) between average velocity and instantaneous velocity?

@ a specific moment in time

Example 10 – Using the Derivative to Find Velocity

At time \( t = 0 \), a diver jumps from a platform diving board that is 32 feet above the water (see Figure 2.21). The position of the diver is given by

\[ s(t) = -16t^2 + 16t + 32 \]  

where \( s \) is measured in feet and \( t \) is measured in seconds.

a) When does the diver hit the water?

b) What is the diver’s velocity at impact?

\[ 0 = -16t^2 + 16t + 32 \]
\[ 0 = -16(t^2 - t - 2) \]
\[ t^2 - t - 2 = 0 \]
\[ (t - 2)(t + 1) = 0 \]
\[ t = 2 \text{ or } t = -1 \]

\( 2 \text{ seconds } \rightarrow \text{ diver hits the water.} \)

\[ s'(t) = -32t + 16 \]
\[ s'(2) = -4 \times 2 + 16 = 8 \text{ ft/sec} \]
0.115 65, 47, 72, 73, 93, 95, 97, 99, 107, 117,